ON THE SUFFICIENT CONDITIONS OF STABILITY OF ROTATION OF A TIPPE-TOP ON A PERFECTLY ROUGH HORIZONTAL SURFACE

(O DOSTATOCHNYKH USLOVIIKH USTOICHIVOSTI VRASHCHENIIA VOLCHKA "TIP-TOP", NAKHODIASHCHEGOSIA NA ABSOLIUTNO SHEROKHOVATOI GORISONTAL'NOI PLOSKOSTI)

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The tippe-top is a top with a spherical base, whose center of gravity is below the center of curvature.

It is assumed that the top is on a perfectly rough plane surface, the friction exerted is dry, and the system is conservative, with an energy integral.

In this paper the author derives equations of motion for a tippe-top on a perfectly rough surface, the ellipsoid of inertia about the center of gravity being an ellipsoid of revolution. The first integrals of the equations of motion are obtained and the conditions of stability of small vibrations of the top axis about the vertical are determined.

Let p, q, r, be projections of the instantaneous value of the angular velocity vector ω on the moving coordinate axes $\xi\eta\zeta$ which coincide with the principal axes of inertia of the top, and whose origin coincides with the center of gravity.

Let γ_1, γ_2, γ_3, be direction cosines with respect to the moving axes $\xi\eta\zeta$ of the unit vector $\mathbf{k}_1$ along the direction of the force of gravity mg, sensed upwards.

Let A be the principal moment of inertia of the top about the axis ξ and the axis η; let C be the principal moment of inertia about the axis ξ; let $\mathbf{k}$ be the unit vector along the axis ζ; let $\mathbf{L}_0$ be the angular momentum of the top about the center of gravity.

The angular momentum of the top about the point of contact of the base is

$$\mathbf{L} = (a\mathbf{k}_1 - l\mathbf{k}) \times \mathbf{Q} + \mathbf{L}_0 \qquad \left(\mathbf{Q} = m\,\frac{d\mathbf{r}}{dt}\right) \tag{1}$$

Here $d\mathbf{r}/dt$ is the velocity of the center of gravity, a is the radius of the sphere, l is the distance from the center of the sphere to the center of gravity.

From (1) we obtain

$$\frac{d\mathbf{L}}{dt} = (a\mathbf{k}_1 - l\mathbf{k}) \times m\,\frac{d^2\mathbf{r}}{dt^2} + \frac{d\mathbf{L}_0}{dt}$$

On the strength of the theorem on angular momentum about the point of contact we have

$$\frac{d\mathbf{L}_0}{dt} + (a\mathbf{k}_1 - l\mathbf{k}) \times m\,\frac{d^2\mathbf{r}}{dt^2} = mg\,(a\mathbf{k}_1 - l\mathbf{k}) \times \mathbf{k}_1 \tag{2}$$

The velocity of the point of contact of the top base is zero

$$\frac{d\mathbf{r}}{dt} - \boldsymbol{\omega} \times (a\mathbf{k}_1 - l\mathbf{k}) = 0$$

Let u, v, w, be projections of $d\mathbf{r}/dt$ on the moving axes. Projecting (2) on the moving axes we obtain

$$\begin{aligned}
A\,\frac{dp}{dt} + (C - A)\,qr + ma\gamma_2\,\frac{dw}{dt} - m\,(a\gamma_3 - l)\,\frac{dv}{dt} &= mgl\gamma_2 \\
A\,\frac{dq}{dt} + (A - C)\,pr + m\,(a\gamma_3 - l)\,\frac{du}{dt} - ma\gamma_1\,\frac{dw}{dt} &= -\,mgl\gamma_1 \\
C\,\frac{dr}{dt} + ma\gamma_1\,\frac{dv}{dt} - ma\gamma_2\,\frac{du}{dt} &= 0
\end{aligned} \tag{3}$$

where

$$w = a\,(\gamma_2 p - \gamma_1 q), \qquad v = a\gamma_1 r - (a\gamma_3 - l)\,p, \qquad u = (a\gamma_3 - l)\,q - a\gamma_2 r \tag{4}$$

Since the direction of the force of gravity is a constant

$$\frac{d\mathbf{k}_1}{dt} = \frac{d'\mathbf{k}_1}{dt} + \boldsymbol{\omega} \times \mathbf{k}_1 = 0 \tag{5}$$

Here the prime denotes the apparent derivative (derivative with respect to moving axes). Projecting (5) on the moving axes we obtain the Poisson equations

$$\frac{d\gamma_1}{dt} = r\gamma_2 - q\gamma_3, \qquad \frac{d\gamma_2}{dt} = p\gamma_3 - r\gamma_1, \qquad \frac{d\gamma_3}{dt} = q\gamma_1 - p\gamma_2 \tag{6}$$

By the kinetic energy theorem we have

$$dT = d\left[\frac{1}{2}\,\mathbf{L}_0 \cdot \boldsymbol{\omega} + \frac{1}{2}\,m\left(\frac{dr}{dt}\right)^2\right] = -\,mgd\,(l\mathbf{k} \cdot \mathbf{k}_1) \tag{7}$$

From (7) we obtain the energy integral

$$\mathbf{L}_0\cdot\boldsymbol{\omega} + m\left(\frac{dr}{dt}\right)^2 + 2mgl\mathbf{k}\cdot\mathbf{k}_1 = 2h = \text{const}$$

or

$$[A + m(a^2 - 2al\gamma_3 + l^2)](p^2+q^2) + [C + ma(a - 2l\gamma_3)]r^2 - \\ - ma^2\omega_1{}^2 + 2malr\omega_1 + 2mgl\gamma_3 = 2h$$

where

$$\omega_1 = p\gamma_1 + q\gamma_2 + r\gamma_3$$

In order to obtain a second integral we dot-multiply (2) by $(a\mathbf{k}_1 - l\mathbf{k})$; since

$\boldsymbol{\omega} = (\mathbf{k}\times\boldsymbol{\omega})\times\mathbf{k} + (\mathbf{k}\cdot\boldsymbol{\omega})\,\mathbf{k}$, a $\mathbf{L}_0 = A\boldsymbol{\omega}' + C\boldsymbol{\omega}''$, $\boldsymbol{\omega}' = (\mathbf{k}\times\boldsymbol{\omega})\times\mathbf{k}$, $\boldsymbol{\omega}'' = (\mathbf{k}\cdot\boldsymbol{\omega})\,\mathbf{k}$, TO

it follows that

$$(\boldsymbol{\omega}\times\mathbf{k})\cdot\mathbf{L}_0 = 0$$

and it means that

$$\frac{d}{dt}[\mathbf{L}_0\cdot(a\mathbf{k}_1 - l\mathbf{k})] = 0$$

Hence

$$\mathbf{L}_0\cdot(a\mathbf{k}_1 - l\mathbf{k}) = P_0 = \text{const}$$

or

$$Aa\omega_1 + (C - A)\,a\gamma_3 r - Clr = P_0 \tag{8}$$

It is seen from the integral (8) that the projection of the vector of angular momentum about the center of gravity on the radius vector from the center of gravity to the point of contact is a constant. In order to obtain a third integral we substitute in (2) the expression for $d^2\mathbf{r}/dt^2$, dot-multiply the result by $\mathbf{k}$, and take into account that $d\mathbf{k}/dt$ is perpendicular to $\mathbf{L}_0$. The result is

$$[C + ma^2(1-\gamma_3{}^2)]\frac{dr}{dt} - ma(a\gamma_3 - l)\left(\gamma_1\frac{dp}{dt} + \gamma_2\frac{dq}{dt}\right) - malr(\gamma_1 q - \gamma_2 p) = 0$$

Substituting $d\gamma_3/dt$ from (6) for $q\gamma_1 - p\gamma_2$ in the above equations, and introducing ω_1 we obtain

$$(C + ma^2)\frac{dr}{dt} - ma(a\gamma_3 - l)\frac{d\omega_1}{dt} - mal\frac{d}{dt}(r\gamma_3) = 0 \tag{9}$$

Substituting in (9) the expression for $d(r\gamma_3)/dt$ from (8), then replacing the expression $(C - A)a\gamma_3 - Cl$ by its equivalent from (8), and integrating we obtain

$$[(C-A)(C+ma^2)-Cml^2]\,r^2-2maP_0\omega_1+Ama^2\omega_1^2=Q_0=\text{const} \tag{10}$$

It is seen from the integral (10) that the projection of the angular momentum about the point of contact on a principal axis of inertia is a constant. Integration of equations (6) gives

$$\gamma_1^2+\gamma_2^2+\gamma_3^2=1$$

Let us consider the stability of rotation of the top about its vertical axis ζ, that is the stability of motion given by the particular solution of the equations of motion (3) and (6)

$$u=v=w=0,\quad p=q=0,\quad r=r_0=\text{const},\quad \gamma_1=\gamma_2=0,\quad \gamma_3=1$$

We are investigating the stability of rotation of the top with respect to the variables p, q, r, γ_1, γ_2, γ_3, under the conditions (4).

In the perturbed motion we put

$$p=\xi_1,\quad q=\xi_2,\quad r=r_0+\xi_3,\quad \gamma_1=\eta_1,\quad \gamma_2=\eta_2,\ \gamma_3=1+\eta_3 \tag{11}$$

where ξ_1, ξ_2, ξ_3, η_1, η_2, η_3, are small quantities.

Since $\omega_1 = p\gamma_1 + q\gamma_2 + r\gamma_3$, it follows that in the perturbed motion

$$\omega_1=r_0+\xi_4,\qquad \xi_4=\xi_1\eta_1+\xi_2\eta_2+\xi_3\eta_3+r_0\eta_3+\xi_3$$

The equations of the perturbed motion are obtained from (3) and (6) by removing from them u, v, w, substituting for p, q, r, γ_1, γ_2, γ_3, the expressions (11) and making $\omega_1 = r_0 + \xi_4$.

The first integrals of the perturbed motion are

$$\begin{gathered}
V_1=\{A+m\,[a^2-2al\,(1+\eta_3)+l^2]\}\,(\xi_1^2+\xi_2^2)+2mgl\eta_3+2mal\xi_4\,(r_0+\xi_3)+\\
+\{C+ma\,[a-l\,(1+2\eta_3)]\}\,(2r_0\xi_3+\xi_3^2)-ma^2\xi_4\,(2r_0+\xi_4)=\text{const}\\
V_2=Aa\xi_4+(C-A)\,ar_0\eta_3+[C\,(a-l)-Aa]\,\xi_3+(C-A)\,a\xi_3\eta_3=\text{const}\\
V_3=[(C-A)\,(C+ma^2)-Cml^2]\,\xi_3^2+2r_0\,[(C-A)\,(C+ma^2)-Cml^2]\,\xi_3-\\
-2mar_0\,[C\,(a-l)-Aa]\,\xi_4+Ama^2\xi_4^2=\text{const}\\
V_4=\eta_1^2+\eta_2^2+2\eta_3+\eta_3^2=0
\end{gathered}$$

The Liapunov function is constructed by the Chetaev method in the form of a combination of the integrals

$$V=V_1+\lambda V_2+\mu V_3+\nu V_4+\rho V_4^2$$

The constants λ and μ are determined by equating to zero all the linear terms of V, with the exception of the positive one $(-2ma^2r_0^2\eta_3)$

$$\lambda=2\,\frac{(mgl+\nu)\,[C-A+ml\,(a-l)]-mar_0^2\,[a\,(A-C)+Al-ml^2\,(a-l)]}{ar_0^2\,(A-C)\,[C+ma\,(a-l)]}$$

$$\mu=\frac{Car_0^2-(a-l)\,[ml\,(g+ar_0^2)+\nu]}{ar_0^2\,(A-C)\,[C+ma\,(a-l)]}$$

The constants ρ and ν are arbitrary, with ρ being a negative or a sufficiently small positive quantity. Let

$$V^* = A_1(\xi_1^2 + \xi_2^2) + A_2\xi_3^2 + 2A_3\xi_3\eta_3 + A_4\eta_3^2 + \nu(\eta_1^2 + \eta_2^2) + A_5(\xi_1\eta_1 + \xi_2\eta_2) + \\ + \rho(\eta_1^2 + \eta_2^2 + \eta_3^2)(\eta_1^2 + \eta_2^2 + \eta_3^2 + 4\eta_3) + ma^2(1 + \mu A)[\xi_1^2\eta_1^2 + \xi_2^2\eta_2^2 + \\ + \xi_3^2\eta_3^2 + 2(\xi_1\xi_2\eta_1\eta_2 + \xi_1\xi_3\eta_1\eta_3 + \xi_1\xi_3\eta_1 + r_0\xi_1\eta_1\eta_3 + \xi_2\xi_3\eta_2\eta_3 + \xi_2\xi_3\eta_2 + r_0\xi_2\eta_2\eta_3 + \\ + \xi_3^2\eta_3 + r_0\xi_3\eta_3^2 + r_0\xi_3\eta_3)] + 2mal(\xi_1\xi_3\eta_1 + \xi_2\xi_3\eta_2 + \xi_3^2\eta_3) - 2ma^2r_0^2\eta_3$$

$$V^* \leqslant V$$

where

$$\begin{aligned} A_1 &= A + m(a-l)^2 \\ A_2 &= C + 2ma^2 + \mu C[C - A + m(a^2 - l^2)] \\ 2A_3 &= Ca[\lambda - 2\mu m r_0(a-l)] \\ A_4 &= r_0^2 ma^2(1 + \mu A) + \nu + 4\rho \\ A_5 &= a[-2mr_0(a-l)(1 + \mu C) + A(2\mu m r_0 a + \lambda)] \end{aligned}$$

The function V is then a positive-definite function of the variables ξ_1, ξ_2, ξ_3, η_1, η_2, η_3, if the quadratic form of the above variables, is also positive-definite

$$V^{**} = A_1(\xi_1^2 + \xi_2^2) + A_2\xi_3^2 + 2A_3\xi_3\eta_3 + A_4\eta_3^2 + \nu(\eta_1^2 + \eta_2^2) + A_5(\xi_1\eta_1 + \xi_2\eta_2)$$

According to Sylvester the necessary and sufficient conditions for positive-definiteness of V^{**} are

$$1)\ A_2 > 0, \quad 2)\ 4A_1\nu - A_5^2 > 0, \quad 3)\ A_2A_4 - A_3^2 > 0, \quad 4)\ \nu > 0, \quad 6)\ A_4 > 0$$

or

$$\begin{aligned} &1) \quad C + 2ma^2 + \mu C[C - A + m(a^2 - l^2)] > 0 \\ &2) \quad 4[A + m(a-l)^2]\nu - a^2[-2mar_0(a-l)(1 + C\mu) + A(2\mu m r_0 a + \lambda)]^2 > 0 \quad (12) \\ &3) \quad 4\{C + 2ma^2 + \mu C[C - A + m(a^2 - l^2)]\}[r_0^2 ma^2(1 + \mu A) + \nu + 4\rho] - \\ &\qquad - C^2a^2[\lambda - 2\mu m r_0(a-l)]^2 > 0 \\ &4) \quad r_0^2 ma^2(1 + \mu A) + \nu + 4\rho > 0 \\ &5) \quad \nu > 0 \end{aligned}$$

According to Liapunov the conditions (12) are the necessary conditions for the stability of rotation of a top about the vertical axis. The conditions (12) become:

if $\mu = 0$

$$\begin{aligned} &1) \quad 4[A + m(a-l)^2]\nu - a^2[-2mr_0(a-l) + \lambda A]^2 > 0 \\ &2) \quad 4(C + 2ma^2)(r_0^2 ma^2 + \nu + 4\rho) - C^2a^2\lambda^2 > 0 \\ &3) \quad r_0^2 ma^2 + \nu + 4\rho > 0 \\ &4) \quad r_0^2 > \frac{(a-l)mgl}{a[C - (a-l)ml]} \end{aligned}$$

where

$$\nu = \frac{Car_0^2 - (a-l)(g + ar_0^2)ml}{a-l}, \qquad \lambda = -\frac{2r_0}{a-l}$$

if $\mu = 0$ and $\rho \gtrless 0$

$$1)\quad 4\,[A + m\,(a-l)^2]\,\nu - a^2\,[-2mr_0\,(a-l) + \lambda A]^2 > 0$$
$$2)\quad 4\,(C + 2ma^2)\,(r_0^2 + ma^2 + \nu) - C^2a^2\lambda^2 > 0$$
$$3)\quad r_0^2 > \frac{(a-l)\,mgl}{a\,[C-(a-l)\,ml]}$$

where

$$\nu = \frac{Car_0^2 - (a-l)\,(g + ar_0^2)\,ml}{a-l}, \qquad \lambda = -\frac{2r_0}{a-l}$$

if $\lambda = 0$

$$1)\quad C + 2ma^2 + C\mu\,[C - A + m\,(a^2 - l^2)] > 0$$
$$2)\quad 4\,[A + m\,(a-l)^2]\,\nu - a^2\,[-2mr_0\,(a-l)\,(1 + C\mu) + 2\mu Amar_0]^2 > 0$$
$$3)\quad \{C + 2ma^2 + C\mu\,[G - A + m\,(a^2 - l^2)]\}\,[r_0^2ma^2\,(1 + \mu A) + \nu + 4\rho] - C^2a^2m^2\mu^2r_0^2\,(a-l)^2 > 0$$
$$4)\quad r_0^2ma^2\,(1 + \mu A) + \nu + 4\rho > 0$$
$$5)\quad r_0^2 > \frac{gl\,[C - A + ml\,(a-l)]}{a\,[(A-C)\,a + Al - ml^2\,(a-l)]}$$

where

$$\mu = -\frac{1}{C - A + ml\,(a-l)}, \qquad \nu = \frac{mar_0^2\,[(A-C)\,a + Al - ml^2\,(a-l)]}{C - A + ml\,(a-l)} - mgl$$

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